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FOR MULTI-ISSUE ALLOCATION SITUATIONS**

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A composite run-to-the-bank rule for multi-issue allocation situations*

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Abstract

In this paper, we propose a new extension of the run-to-the-bank rule for bankruptcy situations to the class of multi-issue allocation situations. We show that this rule always yields a core element and that it satisfies self-duality. We characterise our rule by means of a new consistency property, issue-consistency.

Key words: cooperative games, multi-issue allocation, bankruptcy, self-duality, consistency.

1 Introduction

In a bankruptcy situation (O'Neill (1982)), one has to divide a given amount of money (estate) amongst a set of agents, each of whom has a claim on the estate. The total amount claimed typically exceeds the estate available, so

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not all the claims of the agents can be fully satisfied. Calleja et al. (2001) extend this model to encompass situations in which the agents can have multiple claims on the estate, each as a result of a particular issue. For such multi-issue allocation situations they propose an extension of the run-to-the-bank rule of O'Neill as solution for this new class of problems. As is the case for the original rule, this extended run-to-the-bank rule turns out to coincide with the Shapley value of the corresponding multi-issue allocation game.

Contrary to bankruptcy games, however, multi-issue allocation games need not be convex. Consequently, there exist multi-issue allocation situations for which the run-to-the-bank solution is not a core element of the corresponding game. In this paper, we extend the run-to-the-bank rule in a different way, such that it always yields a core element.

Instead of considering the issues and the players combined, as in Calleja et al. (2001), we propose a two-stage extension: first, we explicitly allocate the estate to the issues (according to a marginal vector), and then, within each issue the money is divided among the agents using the standard run-to-the-bank rule. An alternative view on composite solution is given in Casas-Méndez et al. (2002).

Based on Aumann and Maschler (1985), we define the concept of (self-) duality for multi-issue allocation situations and show that both extensions of the run-to-the-bank-rule are self-dual. Finally, we characterise our composite extension by means of the property of issue-consistency, which generalises the consistency property that was first used by O'Neill (1982).

This paper is organised as follows. In section 2, we present the bankruptcy and multi-issue allocation models and define the bankruptcy run-to-the-bank rule. In section 3, we define our composite extension of this rule and show that this yields a core element. In section 4, we define self-duality and prove that both extensions of the run-to-the-bank rule satisfy this property. Finally, in section 5, we characterise the composite run-to-the-bank rule by means of issue-consistency and we show that this rule is monotonic.

2 Multi-issue allocation situations

A *bankruptcy situation* (O'Neill (1982)) is a triple (N, E, c) , where $N = \{1, \dots, n\}$ is the set of players, $E \geq 0$ is the estate under contest and $c \in \mathbb{R}_+^N$ is the vector of claims such that $\sum_{i \in N} c_i \geq E$.

With each bankruptcy situation (N, E, c) a *bankruptcy game* can be associated with set of players N and characteristic function $v_{E,c}$, which assigns

to each coalition $S \subset N$ the part of the estate that is left for the players in S after the claims of the other players have been satisfied, ie,

$$v_{E,c}(S) = \max\{0, E - \sum_{i \in N \setminus S} c_i\}$$

for all $S \subset N$.

A *multi-issue allocation (MIA) situation* (Calleja et al. (2001)) is a quadruplet (N, R, E, C) , where $N = \{1, \dots, n\}$ is the set of players, $R = \{1, \dots, r\}$ is the set of issues, $E \geq 0$ is the estate and $C \in \mathbb{R}_+^{R \times N}$ is the matrix of claims. We assume that $\sum_{k \in R, i \in N} c_{ki} \geq E$, $\sum_{k \in R} c_{ki} > 0$ for all $i \in N$ and $\sum_{i \in N} c_{ki} > 0$ for all $k \in R$.

Given a matrix C , we denote by C_i the i th row of C , and by C_{-i} the matrix C without the i th row. Furthermore, we denote $c_{kS} = \sum_{i \in S} c_{ki}$ for $S \subset N$ and $c_{Ki} = \sum_{k \in K} c_{ki}$ for $K \subset R$. In this way the sum of the components of C_i is denoted by c_{iN} , and the total claim of player $i \in N$ is c_{Ri} .

A permutation τ on R is a bijection $\tau : \{1, \dots, r\} \rightarrow R$, where $\tau(p)$ denotes which element of R is at position p . The set of all $r!$ permutations on R is denoted by $\Pi(R)$. The reverse permutation of τ , $\tau_{\text{rev}} \in \Pi(R)$ is defined by $\tau_{\text{rev}}(p) = \tau(n + 1 - p)$ for all $p \in \{1, \dots, r\}$.

For a MIA situation (N, R, E, C) , we define a corresponding *MIA game* by assigning to each coalition S the minimum amount they can guarantee themselves if the players in $N \setminus S$ are free to choose an order on the issues and the players, where we assume that an issue cannot be dealt with until the previous one is completed. Given an order on the issues $\tau \in \Pi(R)$, we define the index t_τ by

$$t_\tau = \max\{t \mid \sum_{p=1}^t c_{\tau(p)N} \leq E\}. \quad (1)$$

So, the issues $\tau(1), \dots, \tau(t_\tau)$ will be entirely satisfied for all the players, whereas the issue $\tau(t_\tau + 1)$ will only be partially satisfied, with the amount $E_\tau = E - \sum_{p=1}^{t_\tau} c_{\tau(p)N}$. The remaining issues are not handled at all according to τ . In this way, the amount $f_S(\tau)$ which coalition S gets at least equals

$$f_S(\tau) = \sum_{p=1}^{t_\tau} c_{\tau(p)S} + \max\{0, E_\tau - c_{\tau(t_\tau+1)N \setminus S}\}.$$

Then the MIA game is given by the function

$$v_{E,C}(S) = \min_{\tau \in \Pi(R)} f_S(\tau)$$

for all $S \subset N$. Note that this MIA game corresponds to the Q-approach in Calleja et al. (2001).

A bankruptcy situation (N, E, c) can be viewed as a MIA situation (N, R, E, C) in two ways:

- $|R| = 1$ and $C = c$,
- $R = N$ and $C = \text{diag}(c)$, ie, the claim matrix is the diagonal matrix with the elements of c on the diagonal.

A *bankruptcy rule* is a function f assigning to every bankruptcy situation (N, E, c) a vector $f(N, E, c) \in \mathbb{R}^N$ such that:

- i) $0 \leq f_i(N, E, c) \leq c_i$ for all $i \in N$,
- ii) $\sum_{i \in N} f_i(N, E, c) = E$.

A well-known example of a bankruptcy rule is the run-to-the-bank (RTB) rule, introduced by O'Neill (1982), although under a different name (recursive completion). This rule turns out to coincide with the Shapley value of the corresponding bankruptcy game. In Section 3 we give a definition of this rule.

A *MIA rule* is a function g assigning to every MIA situation (N, R, E, C) a vector $g(N, R, E, C) \in \mathbb{R}^N$ such that

- i) $0 \leq g_i(N, R, E, C) \leq c_{Ri}$ for all $i \in N$,
- ii) $\sum_{i \in N} g_i(N, R, E, C) = E$.

Calleja et al. (2001) extend the RTB rule to the class of MIA situations and show that this RTB rule coincides with the Shapley value of the corresponding MIA game.

3 The composite run-to-the-bank rule

In this section, we extend the RTB rule for bankruptcy situations to the class of MIA situations. Contrary to the extension in Calleja et al. (2001), our rule (*mRTB*) involves multiple runs to the bank, once by the issues and within each issue by the players.

This two-stage procedure is illustrated in the following picture, where f and g are bankruptcy rules:

$$\begin{array}{ccc}
(N, R, E, C) & \xrightarrow{\text{composite rule}} & s \in \mathbb{R}_+^N \\
\downarrow & & \uparrow \\
(R, E, (c_{kN})_{k \in R}) & \xrightarrow{f} s^0 \in \mathbb{R}_+^R \longrightarrow & \left\{ \begin{array}{c} (N, s_1^0, C_1) \xrightarrow{g} s^1 \in \mathbb{R}_+^N \\ \vdots \\ (N, s_r^0, C_r) \xrightarrow{g} s^r \in \mathbb{R}_+^N \end{array} \right\} +
\end{array}$$

The bankruptcy game corresponding to the situation $(R, E, (c_{kN})_{k \in R})$ will be denoted by $v_{E,C}^R$.

In order to introduce the *mRTB* rule, we first define the *RTB* rule in terms of marginal vectors. Given a cooperative game with player set N and characteristic function v , we define for each permutation $\sigma \in \Pi(N)$ the *marginal vector* $m^\sigma(v)$ by

$$m_{\sigma(p)}^\sigma(v) = v(\{\sigma(1), \dots, \sigma(p)\}) - v(\{\sigma(1), \dots, \sigma(p-1)\})$$

for all $p \in \{1, \dots, n\}$.

The *run-to-the-bank* rule for bankruptcy games, *RTB*, coincides with the Shapley value and can thus be expressed as

$$\text{RTB}(N, E, c) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v_{E,c}).$$

Let (N, R, E, C) be a MIA situation. For $\tau \in \Pi(R)$ and $\sigma \in \Pi(N)$, we define the *composite marginal vector* as

$$mm^{\tau,\sigma}(N, R, E, C) = \sum_{k \in R} m^\sigma(v_{x_k, C_k}),$$

where $x = m^\tau(v_{E,C}^R)$.

The set of all composite marginal vector is a subset of the core of the corresponding MIA game, where the core of a game v is defined by

$$\text{Core}(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \forall S \subset N : \sum_{i \in S} x_i \geq v(S)\}.$$

Proposition 1 *Let (N, R, E, C) be a MIA situation. Then*

$$mm^{\tau,\sigma}(N, R, E, C) \in \text{Core}(v_{E,C})$$

for all $\tau \in \Pi(R)$, $\sigma \in \Pi(N)$.

Proof. Let $\tau \in \Pi(R), \sigma \in \Pi(N)$ and let $z = mm^{\tau, \sigma}(N, R, E, C)$. Let x be the marginal vector $m^{\tau}(v_{E, C}^R)$ and $t = t_{\tau}$ (as defined in (1)). With x_k as estate for issue $k \in R$, we have a collection of bankruptcy situations $\{(N, x_k, C_k)\}_{k \in R}$. However, at most one of them is a nontrivial situation: in the situations $\tau(1), \dots, \tau(t)$ the estate equals the sum of all the claims and in the situations $\tau(t+2), \dots, \tau(r)$ the estate equals zero. Let y be the marginal vector corresponding to σ of the only possible nontrivial bankruptcy situation $(N, x_{\tau(t+1)}, C_{\tau(t+1)})$:

$$y = m^{\sigma}(v_{x_{\tau(t+1)}, C_{\tau(t+1)}}).$$

We can express vector z as

$$z = y + \sum_{p=1}^t C_{\tau(p)}.$$

Given a coalition $S \subset N$ we have defined $v_{E, C}(S)$ as $\min_{\tau \in \Pi(R)} f_S(\tau)$, but $\sum_{i \in S} z_i$ is precisely $f_S(\tau)$ if in the permutation σ the players in S are at the end, and for any other permutation this amount is larger or equal. Hence, $\sum_{i \in S} z_i \geq v_{E, C}(S)$ for all $S \subset N$, and so, z is in the core of $v_{E, C}$. \square

A general relation of inclusion between the set of marginal vectors and the set of composite marginal vectors cannot be established, as is shown the following example.

Example 1 Let (N, R, E, C) be the MIA situation with $N = \{1, 2, 3\}$, $R = \{1, 2\}$, estate $E = 10$ and claim matrix $C = \begin{pmatrix} 9 & 5 & 0 \\ 3 & 7 & 7 \end{pmatrix}$. The game associated with this situation is

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	N
$v_{E, C}(S)$	0	0	0	3	3	1	10

The sets of marginal and composite marginal vectors can be easily calculated. The results are given in the following tables.

$\sigma \in \Pi(N)$	m^{σ}	$\tau \in \Pi(R)$	$\sigma \in \Pi(N)$	$mm^{\tau, \sigma}$
123	(0, 3, 7)	12	123, 132, 312	(9, 1, 0)
132	(0, 7, 3)		213, 231, 321	(5, 5, 0)
213	(3, 0, 7)	21	123, 213	(3, 7, 0)
231	(9, 0, 1)		132, 312	(3, 0, 7)
312	(3, 7, 0)		231	(0, 7, 3)
321	(9, 1, 0)		321	(0, 3, 7)

The tables show that $m^{231}(v_{E,C})$ is not a composite marginal vector and that $mm^{12,213}(N, R, E, C)$ does not belong to the set of marginal vectors of the game $v_{E,C}$. \square

Now we define the m RTB rule, which extends the RTB rule for bankruptcy situations to the class of MIA situations.

Definition 1 *Let (N, R, E, C) be a multi-issue allocation situation. The m RTB rule is defined by*

$$mRTB(N, R, E, C) = \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} RTB(N, m_k^\tau(v_{E,C}^R), C_k). \quad (2)$$

The m RTB rule can be interpreted as the result of two races: first, the issues “run to the bank” for the money, and next, there are r races among the claimants within each issue. As is the case for the RTB rule for bankruptcy situations, the claims are satisfied as much as possible by the order of arrival.

This m RTB rule first takes the marginal vectors of the “issue game” $v_{E,C}^R$. Associated with each marginal vector $m^\tau(v_{E,C}^R)$ we have r bankruptcy games whose estates are given by the components of the marginal vector. Next, we take for each player the sum of the RTB solutions of these r situations. Finally, the average among all the marginals is computed. It is readily seen that the m RTB rule can be expressed as

$$mRTB(N, R, E, C) = \frac{1}{r!} \sum_{\tau \in \Pi(R)} \frac{1}{n!} \sum_{\sigma \in \Pi(N)} mm^{\tau, \sigma}(N, R, E, C). \quad (3)$$

It is easy to see that if we start with a bankruptcy situation (N, E, c) and construct one of the two corresponding MIA situations (N, R, E, C) as indicated in Section 2, then $RTB(N, E, c) = mRTB(N, R, E, C)$. So, the m RTB rule is indeed an extension of the RTB rule. However, the m RTB rule does not in general coincide with the Shapley value of the game. In fact, the m RTB rule is not even game-theoretic, ie, two situations leading to the same game might yield different outcomes.

The m RTB rule provides a very easy way of obtaining an element of the core of a MIA game without calculating the characteristic function. This is stated in the following theorem.

Theorem 1 *Let (N, R, E, C) be a MIA situation. Then*

$$mRTB(N, R, E, C) \in \text{Core}(v_{E,C}).$$

Proof. In Proposition 1, we show that every composite marginal vector lies in the core. The m RTB outcome, being the average of these composite marginals vectors according to equation (3), then also is an element of the core, which is a convex set. \square

As an alternative to the m RTB rule, another way to extend the RTB rule in a two-stage way would be to apply the RTB rule twice: $\sum_{k \in R} \text{RTB}(N, x_k, C_k)$ with $x = \text{RTB}(R, E, (c_{kN})_{k \in R})$. However, this solution can lie outside the core of the corresponding MIA game, as the next example shows.

Example 2 Consider the MIA situation (N, R, E, C) with $N = \{1, 2, 3\}$, $R = \{1, 2, 3\}$, estate $E = 51$ and claim matrix

$$C = \begin{pmatrix} 0 & 2 & 6 \\ 0 & 1 & 24 \\ 24 & 2 & 0 \end{pmatrix}$$

The game associated with this situation is

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	N
$v_{E,C}(S)$	16	3	22	21	46	27	51

We have $x = \text{RTB}(R, E, (c_{kN})_{k \in R}) = (\frac{16}{3}, \frac{67}{3}, \frac{70}{3})$ and $\sum_{k \in R} \text{RTB}(N, x_k, C_k) = (\frac{67}{3}, \frac{5}{2}, \frac{157}{6})$. As $\frac{5}{2} < 3 = v_{E,C}(\{2\})$, this solution is not in the core of $v_{E,C}$. \square

4 Self-duality

For a MIA situation (N, R, E, C) we denote by $D(S) = c_{RS}$, ie, the total claim of the players in coalition S , and we define $D = D(N)$. Recall that we assume $D \geq E$.

Lemma 1 Let (N, R, E, C) be a MIA situation. Then

$$v_{E,C}(S) = v_{D-E,C}(N \setminus S) + D(S) - D + E.$$

Proof. To calculate the value of $v_{E,C}(S)$, we must find a permutation on the players $\sigma \in \Pi(N)$ and a permutation on the issues $\tau \in \Pi(R)$ such that the total amount assigned to coalition S is minimal. Obviously, σ can be any permutation in which the players in S are at the end.

In Figure 1 we represent all the claims of matrix C in the order indicated by τ and σ , ie, $c_{\tau(1)\sigma(1)}, c_{\tau(1)\sigma(2)}, \dots, c_{\tau(r)\sigma(n)}$. The claims associated with players in S are shaded. The total claimed is divided into two parts of

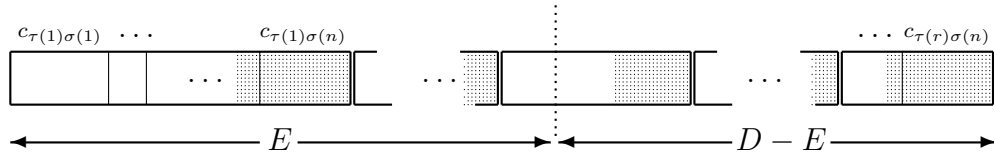


Figure 1: Proof of Lemma 1

lengths E and $D - E$, as the figure shows. From the way in which σ and τ are chosen, the dark zone in the E part is as small as possible, and it is precisely $v_{E,C}(S)$.

If now we consider the MIA situation $(N, R, D - E, C)$ and we want to calculate $v_{D-E,C}(N \setminus S)$, we must find $\sigma' \in \Pi(N)$ and $\tau' \in \Pi(R)$ such that the white zone in the $D - E$ segment is minimised. The length of this zone is indeed $v_{D-E,C}(N \setminus S)$. It is easy to see that this minimum is reached for σ_{rev} and τ_{rev} .

On the other hand, we have that the E segment is the sum of its white and shaded parts. The white part within E will be the total white zone $D(N \setminus S)$ minus the white zone in the $D - E$ segment. The shaded part of E is $v_{E,C}(S)$, as was indicated above. So,

$$E = v_{E,C}(S) + D(N \setminus S) - v_{D-E,C}(N \setminus S).$$

From the equality $D = D(S) + D(N - S)$, we conclude that the statement holds. \square

The next lemma gives us the relation between the marginal vectors of the two MIA games with estates E and $D - E$.

Lemma 2 *The marginal vectors of the games induced by the MIA situations (N, R, E, C) and $(N, R, D - E, C)$ satisfy the following relationship:*

$$m^\sigma(v_{E,C}) = ((c_{Ri})_{i \in N}) - m^{\sigma_{\text{rev}}}(v_{D-E,C})$$

for each $\sigma \in \Pi(N)$.

Proof. Let $\sigma \in \Pi(N)$ and $p \in \{1, \dots, n\}$. Let $i = \sigma(p)$ and let S be the coalition $\{\sigma(1), \dots, \sigma(p-1)\}$. Then

$$m_i^\sigma(v_{E,C}) = v_{E,C}(S \cup \{i\}) - v_{E,C}(S).$$

From Lemma 1 we have that

$$\begin{aligned}
m_i^\sigma(v_{E,C}) &= v_{D-E,C}(N \setminus (S \cup \{i\})) + D(S \cup \{i\}) - D + E \\
&\quad - [v_{D-E,C}(N \setminus S) + D(S) - D + E] \\
&= D(\{i\}) + v_{D-E,C}(N \setminus (S \cup \{i\})) - v_{D-E,C}(N \setminus S) \\
&= D(\{i\}) - m_i^{\sigma_{\text{rev}}}(v_{D-E,C}).
\end{aligned}$$

Since $D(\{i\}) = c_{Ri}$, the result follows. \square

Following Aumann and Maschler (1985), given a rule f we can define its dual f^* by using f to share not the estate E but the gap $D - E$. So, each player receives his claim (the part he would receive if the estate were big enough) minus the corresponding part of the losses:

$$f^*(N, R, E, C) = (c_{Ri})_{i \in N} - f(N, R, D - E, C).$$

A rule is called self-dual if $f^* = f$. We show that both extensions of the RTB rule are self-dual.

Proposition 2 *The RTB rule for MIA situations (cf. Calleja et al. (2001)) is self-dual.*

Proof. Calleja et al. (2001) show that the RTB rule coincides with the Shapley value of the associated cooperative game. So, for a MIA situation (N, R, E, C) ,

$$\text{RTB}(N, R, E, C) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v_{E,C}).$$

From Lemma 2 it then follows that

$$\begin{aligned}
\frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v_{E,C}) &= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} [(c_{Ri})_{i \in N} - m^{\sigma_{\text{rev}}}(v_{D-E,C})] \\
&= (c_{Ri})_{i \in N} - \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(v_{D-E,C}) \\
&= (c_{Ri})_{i \in N} - \text{RTB}(N, R, D - E, C).
\end{aligned}$$

This shows that the RTB rule is self-dual. \square

As a result of the previous proposition, the RTB rule is self-dual for bankruptcy situations as well, which was first proved by Curiel (1988).

Theorem 2 *The mRTB rule is self-dual.*

Proof. Let (N, R, E, C) be a MIA situation. We will denote by v_E^R and v_{D-E}^R the characteristic functions of the games induced by the bankruptcy situations $(N, E, (c_{kN})_{k \in R})$ and $(N, D - E, (c_{kN})_{k \in R})$, respectively. Then, using self-duality of the RTB rule for bankruptcy situations,

$$\begin{aligned}
mRTB(N, R, E, C) &= \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} RTB(N, m_k^\tau(v_E^R), C_k), \\
&= \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} RTB(N, c_{kN} - m_k^{\tau_{\text{rev}}}(v_{D-E}^R), C_k) \\
&= \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} [C_k - RTB(N, c_{kN} - c_{kN} + m_k^{\tau_{\text{rev}}}(v_{D-E}^R), C_k)] \\
&= \sum_{k \in R} C_k - \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} RTB(N, m_k^\tau(v_{D-E}^R), C_k) \\
&= (c_{Ri})_{i \in N} - mRTB(N, R, D - E, C).
\end{aligned}$$

Hence, the $mRTB$ rule is self-dual. \square

5 Issue consistency and monotonicity

In this section we characterise the $mRTB$ rule as a consistent extension of the RTB rule for bankruptcy situations to multi-issue allocations situations. This so-called issue-consistency allows us to establish monotonicity of the $mRTB$ rule.

Definition 2 *A bankruptcy rule f is called claim-consistent (cf. O'Neill (1982)) if for each bankruptcy situation (N, E, c) the following relation holds:*

$$f_i(N, E, c) = \frac{1}{n} \left[\min\{E, c_i\} + \sum_{j \in N \setminus \{i\}} f_i(N \setminus \{j\}, \max\{E - c_j, 0\}, (c_t)_{t \in N \setminus \{j\}}) \right] \quad (4)$$

for all $i \in N$.

If a rule is claim-consistent, the solution can be viewed as an average of n payoffs. Each payoff is calculated by fixing a player $j \in N$ and giving him as

much as possible, $\min\{E, c_j\}$; then, the remaining $\max\{E - c_j, 0\}$ is shared among the other players.

This property determines a unique rule for bankruptcy situations: the recursive completion method of O'Neill (1982), which coincides with the RTB rule and, hence, with the Shapley value of the associated game.

For MIA situations we define a new kind of consistency. A rule is issue-consistent if it can be expressed as an average of payoffs too, but now the payoffs are calculated by fixing an issue $k \in R$ and allocating to it the amount $\min\{E, c_{iN}\}$, while the remaining estate is shared among the remaining issues.

Definition 3 *A MIA rule f is called issue-consistent if for each MIA situation (N, R, E, C) the following relation holds:*

$$f(N, R, E, C) = \frac{1}{r} \sum_{k \in R} \left[f(N, \{k\}, \min\{E, c_{kN}\}, C_k) + f(N, R \setminus \{k\}, \max\{E - c_{kN}, 0\}, C_{-k}) \right]. \quad (5)$$

Issue-consistency allows us to extend any rule defined for bankruptcy situations to MIA situations: the first term of the summation in (5) applies the rule f to a (perhaps trivial) bankruptcy situation, while the second term applies f to a MIA situation with $r - 1$ issues, so the expression can be recursively expanded until f is used only on bankruptcy situations (ie, MIA situations with only one issue). Analogous to claim-consistency, every bankruptcy rule has a unique issue-consistent extension.

Theorem 3 *The mRTB rule is the issue-consistent extension of the RTB rule.*

Proof. Let (N, R, E, C) be a MIA situation. Then

$$\gamma = m\text{RTB}(N, R, E, C) = \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R} \text{RTB}(N, m_k^\tau(v_{E,C}^R), C_k).$$

For each permutation τ on the issues we split the second summation into one term corresponding to its first element $\tau(1)$, and the terms associated with the remaining terms:

$$\gamma = \frac{1}{r!} \sum_{\tau \in \Pi(R)} \left[\text{RTB}(N, m_{\tau(1)}^\tau(v_{E,C}^R), C_{\tau(1)}) + \sum_{k \in R \setminus \{\tau(1)\}} \text{RTB}(N, m_k^\tau(v_{E,C}^R), C_k) \right].$$

Since $m_{\tau(1)}^\tau(v_{E,C}^R) = \min\{E, c_{\tau(1)N}\}$ and there are $(r-1)!$ permutations τ in $\Pi(R)$ with $\tau(1) = i$, we can write

$$\begin{aligned} \gamma = \frac{1}{r!} (r-1)! \sum_{k \in R} \text{RTB}(N, \min\{E, c_{kN}\}, C_k) \\ + \frac{1}{r!} \sum_{\tau \in \Pi(R)} \sum_{k \in R \setminus \{\tau(1)\}} \text{RTB}(N, m_k^\tau(v_{E,C}^R), C_k). \end{aligned}$$

The $m\text{RTB}$ rule coincides with RTB if there is only one issue, ie,

$$\text{RTB}(N, \min\{E, c_{kN}\}, C_k) = m\text{RTB}(N, \{k\}, \min\{E, c_{kN}\}, C_k).$$

Denoting by $v^{R \setminus \{k\}}$ the bankruptcy game associated with the situation $(R, \max\{E - c_{kN}, 0\}, (c_{\ell N})_{\ell \in R \setminus \{k\}})$ we have

$$\begin{aligned} \gamma = \frac{1}{r} \sum_{k \in R} m\text{RTB}(N, \{k\}, \min\{E, c_{kN}\}, C_k) \\ + \frac{1}{r!} \sum_{k \in R} \left[\sum_{\tau \in \Pi(R \setminus \{k\})} \sum_{\ell \in R \setminus \{k\}} \text{RTB}(N, m_\ell^\tau(v^{R \setminus \{k\}}), C_\ell) \right]. \end{aligned}$$

Comparing the expression in brackets with the definition of $m\text{RTB}$ given in equation (2) yields

$$\begin{aligned} m\text{RTB}(N, R, E, C) = \frac{1}{r} \sum_{k \in R} m\text{RTB}(N, \{k\}, \min\{E, c_{kN}\}, C_k) \\ + \frac{1}{r} \sum_{k \in R} m\text{RTB}(N, R \setminus \{k\}, \max\{E - c_{kN}, 0\}, C_{-k}). \end{aligned}$$

This shows that the $m\text{RTB}$ rule is issue-consistent. This, together with the uniqueness of issue-consistent extension, prove the result. \square

Issue-consistency allows us to show that the $m\text{RTB}$ rule is monotonic. A rule is called monotonic if no player gets less when the estate increase.

Definition 4 *A MIA rule f is monotonic if for every pair of MIA situations (N, R, E, C) and (N, R, E', C) with $E' \geq E$ we have that*

$$f_i(N, R, E', C) \geq f_i(N, R, E, C)$$

for all $i \in N$.

Theorem 4 *The mRTB rule is monotonic.*

Proof. We show that mRTB rule is monotonic by induction on the number of issues r . If $r = 1$ then mRTB coincides with RTB and this rule is monotonic on the class of bankruptcy games (Curiel (1988)).

Next, assume that mRTB is monotonic for situations with $r - 1$ issues. Let C be a claim matrix with r rows. By issue-consistency we have

$$mRTB(N, R, E, C) = \frac{1}{r} \sum_{k \in R} \left[mRTB(N, \{k\}, \min\{E, c_{kN}\}, C_k) + mRTB(N, R \setminus \{k\}, \max\{E - c_{kN}, 0\}, C_{-k}) \right].$$

In the first term inside the brackets we actually apply the RTB rule to a bankruptcy situation. So, by monotonicity of the RTB rule, this term increases if the estate is raised. The second term is the application of mRTB to a $(r - 1)$ -issue allocation situation, which by the induction hypothesis satisfies the monotonicity property. Adding up all terms, we have that mRTB is monotonic. \square

References

- Aumann, R. and M. Maschler (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory*, **36**, 195–213.
- Calleja, P., P. Borm, and R. Hendrickx (2001). Multi-issue allocation games. CentER Discussion Paper 2001–30, Tilburg University, Tilburg, The Netherlands.
- Casas-Méndez, B., P. Borm, L. Carpenté, and R. Hendrickx (2002). The constrained equal award rule for bankruptcy situations with a priori unions. CentER Discussion Paper 2002–83, Tilburg University, Tilburg, The Netherlands.
- Curiel, I. (1988). *Cooperative game theory and applications*. Ph. D. thesis, Nijmegen University, Nijmegen, The Netherlands.
- O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences*, **2**, 345–371.